CS256/Winter 2009 Lecture #5

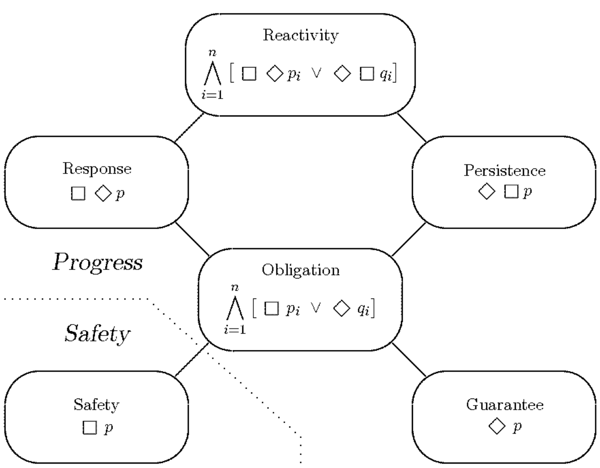
Zohar Manna

Classification Diagram (Fig. 0.18)

- For each $\kappa \in \{$safety, guarantee, obligation response, persistence, reactivity$\}$ the κ class of temporal formulas is characterized by a canonical κ-formula, with p, q, p_i, q_i – past formulas

- A formula is a κ-formula if it is equivalent to a canonical κ-formula

- A property is a κ-property if it is specifiable by a κ-formula



Closure of Classes

Reactivity: closure under $\wedge, \vee, \neg$

Persistence: closure under $\wedge, \vee$

$\Diamond \Box p \wedge \Diamond \Box q \;\sim\; \Diamond \Box (p \wedge q)$

$\Diamond \Box p \vee \Diamond \Box q \;\sim\; \Diamond \Box (q \vee \ominus(p \mathcal{S}(p \wedge \neg q)))$

Response: closure under $\wedge, \vee$

$\Box \Diamond p \vee \Box \Diamond q \;\sim\; \Box \Diamond (p \vee q)$

$\Box \Diamond p \wedge \Box \Diamond q \;\sim\; \Box \Diamond (q \wedge \ominus((\neg q) \mathcal{S} p))$

Obligation: closure under $\wedge, \vee, \neg$

Guarantee: closure under $\wedge, \vee$

$\Diamond p \vee \Diamond q \;\sim\; \Diamond (p \vee q)$

$\Diamond p \wedge \Diamond q \;\sim\; \Diamond (\diamondminus p \wedge \diamondminus q)$

Safety: closure under $\wedge, \vee$

$\Box p \wedge \Box q \;\sim\; \Box (p \wedge q)$

$\Box p \vee \Box q \;\sim\; \Box (\boxminus p \vee \boxminus q)$

Duality of classes

- Safety vs. Guarantee

$\neg \Box p \;\sim\; \Diamond \neg p$
$\neg \Diamond p \;\sim\; \Box \neg p$

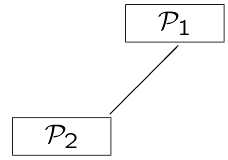
- Response vs. Persistence

$\neg \Box \Diamond p \;\sim\; \Diamond \Box \neg p$
$\neg \Diamond \Box p \;\sim\; \Box \Diamond \neg p$

Classification Diagram

- strict inclusion between boxes

$\mathcal{P}_1 \supset \mathcal{P}_2$



Example: Obligation $\subset$ Persistence
$(\Box p_i \vee \Diamond q_i) \;\sim\; \Diamond \Box (\boxminus p_i \vee \diamondminus q_i)$

> `Theorem:` Every *quantifier free temporal* formula is equivalent to a *reactivity* formula.

Classification Diagram Con't

- strict inclusion between conjunctions
  (Obligation and Reactivity)

In Obligation

$$\bigwedge_{i=1}^{n+1} [\Box p_i \vee \Diamond q_i] \quad \supset \quad \bigwedge_{i=1}^{n} [\Box p_i \vee \Diamond q_i]$$

In Reactivity

$$\bigwedge_{i=1}^{n+1} [\Box \Diamond p_i \vee \Diamond \Box q_i] \quad \supset \quad \bigwedge_{i=1}^{n} [\Box \Diamond p_i \vee \Diamond \Box q_i]$$

Note:

Properties specified by state formulas are safety properties and guarantee properties, since

$$p \;\sim\; \Box(first \rightarrow p)$$

$$p \;\sim\; \Diamond(first \wedge p)$$

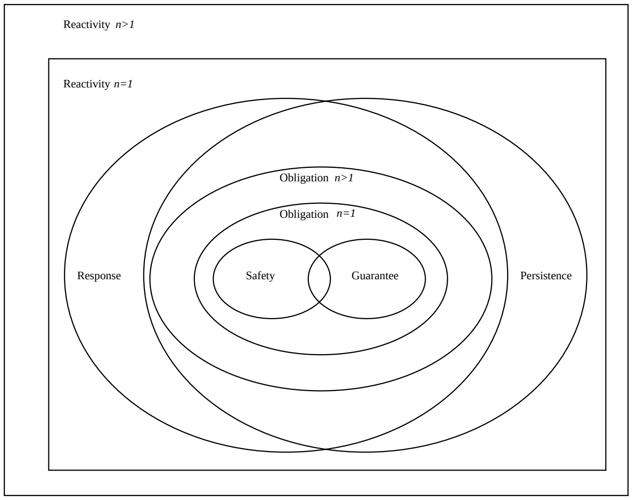
but also $\bigcirc p, \bigcirc \bigcirc p, \ldots$ since

$$\bigcirc p \;\sim\; \Box(\ominus\, first \;\rightarrow\; p)$$

$$\bigcirc p \;\sim\; \Diamond(\ominus\, first \;\wedge\; p)$$

$$\bigcirc \bigcirc p \;\sim \Box(\ominus \ominus\, first \;\rightarrow p)$$

$$\bigcirc \bigcirc p \;\sim \Diamond(\ominus \ominus\, first \wedge p)$$



Example Formulas

- <u>Safety</u> $\Box\, p$

  conditional safety

  $$p \rightarrow \Box\, q \quad \sim \quad \Box(\diamondminus(p \wedge first) \rightarrow \ q)$$
  $$p \Rightarrow \Box\, q \quad \sim \quad \Box(\diamondminus\, p \ \rightarrow \ q)$$

  waiting-for

  $$p\,\mathcal{W}\,q \quad \sim \quad \Box(\diamondminus \neg p \ \rightarrow \ \diamondminus\, q)$$

- <u>Guarantee</u> $\Diamond\, p$

  conditional guarantee

  $$p \rightarrow \Diamond\, q \quad \sim \quad \Diamond\Big(\diamondminus(\mathit{first} \wedge p) \rightarrow q\Big)$$

  until

  $$p\,\mathcal{U}\,q \quad \sim \quad \Diamond(q \wedge \widehat{\boxminus} p)$$

Example formulas (Con't)

- <u>Obligation</u> $\bigwedge_{i=1}^{n+1} (\Box\, p_i \vee \Diamond\, q_i)$

  $$p\,\mathcal{W}\,(\Diamond\, q) \quad \sim \quad \Box\, p \ \vee \ \Diamond\, q$$

- <u>Response</u> $\Box \Diamond\, p$

  response

  $$p \Rightarrow \Diamond\, q \quad \sim \quad \Box \Diamond\Big((\neg p)\,\mathcal{B}\,q\Big)$$

  justice

  $$\Box \Diamond(\neg enabled(\tau) \vee last\text{-}taken(\tau))$$

  where

  $$enabled(\tau): \quad \exists V' \,.\, \rho_\tau(V, V')$$

Example formulas (Con't)

- <u>Persistence</u> $\Diamond \Box\, p$

  conditional stabilization

  $$p \Rightarrow \Diamond \Box\, q \quad \sim \quad \Diamond \Box(\diamondminus\, p \ \rightarrow \ q)$$

- <u>Reactivity</u> $\bigwedge_{i=1}^{n+1} (\Diamond \Box\, p_i \ \vee \ \Box \Diamond\, q_i)$

  compassion

  $$\Box \Diamond\, enabled(\tau) \rightarrow \Box \Diamond\, last\text{-}taken(\tau)$$

  insistence

  $$\Box \Diamond\, p \Rightarrow \Diamond\, q \quad \sim \quad \Box \Diamond\, q \ \vee \ \Diamond \Box \neg p$$

Temporal Logic and Programs: Examples

Control Predicates

at_ℓ	$[\ell] \in \pi$
$at\_\ell_{i,j}$	$at\_\ell_i \lor at\_\ell_j$
$at\_\ell_{i \ldots j}$	$at\_\ell_i \lor at\_\ell_{i+1} \lor \ldots \lor at\_\ell_j$

Example 1: Program BINOM

Compute the binomial coefficient $\binom{n}{k}$ where $0 \le k \le n$

$$b \;=\; \frac{n \cdot (n-1) \cdots y_1 \cdots (n-k+1)}{1 \cdot \quad 2 \cdot \quad \cdots y_2 \cdots \quad \cdot k}$$

property of integers:

a product of m consecutive integers
is evenly divisible by m!

Program BINOM (Fig. 120)

in	k, n	: **integer**	**where**	$0 \le k \le n$
local	y_1, y_2, r	: **integer**	**where**	$y_1 = n, y_2 = 1, r = 1$
out	b	: **integer**	**where**	$b = 1$

$$P_1 :: \left[\begin{array}{l} \textbf{local } t_1 \colon \textbf{integer} \\ \ell_0 : \ \textbf{while } y_1 > (n-k) \textbf{ do} \\ \qquad \left[\begin{array}{l} \ell_1 : \ \textbf{request}(r) \\ \boxed{\begin{array}{l} \ell_2 : \ t_1 := b \cdot y_1 \\ \ell_3 : \ b := t_1 \end{array}} \\ \ell_4 : \ \textbf{release}(r) \\ \ell_5 : \ y_1 := y_1 - 1 \end{array}\right] \\ \ell_6 : \end{array}\right]$$

$$||$$

$$P_2 :: \left[\begin{array}{l} \textbf{local } t_2 \colon \textbf{integer} \\ m_0 : \ \textbf{while } y_2 \le k \textbf{ do} \\ \qquad \left[\begin{array}{l} m_1 : \ \textbf{await } (y_1 + y_2) \le n \\ m_2 : \ \textbf{request}(r) \\ \boxed{\begin{array}{l} m_3 : \ t_2 := b \textbf{ div } y_2 \\ m_4 : \ b := t_2 \end{array}} \\ m_5 : \ \textbf{release}(r) \\ m_6 : \ y_2 := y_2 + 1 \end{array}\right] \\ m_7 : \end{array}\right]$$

Program BINOM : Total Correctness

- termination

$$\Diamond \, [at\_\ell_6 \ \land \ at\_m_7]$$

- partial correctness

$$\Box \left[at\_\ell_6 \ \land \ at\_m_7 \ \rightarrow \ b = \binom{n}{k}\right]$$

- total correctness

$$\boxed{\Diamond \left[at\_\ell_6 \ \land \ at\_m_7 \ \land \ b = \binom{n}{k}\right]}$$

Program BINOM : Auxiliary Properties

- global invariant

$$\square[(n-k) \le y_1 \le n \;\wedge\; 1 \le y_2 \le k+1]$$

- deadlock freedom

$$\square[[at\_\ell_1 \;\wedge\; at\_m_2] \;\rightarrow\; r=1]$$

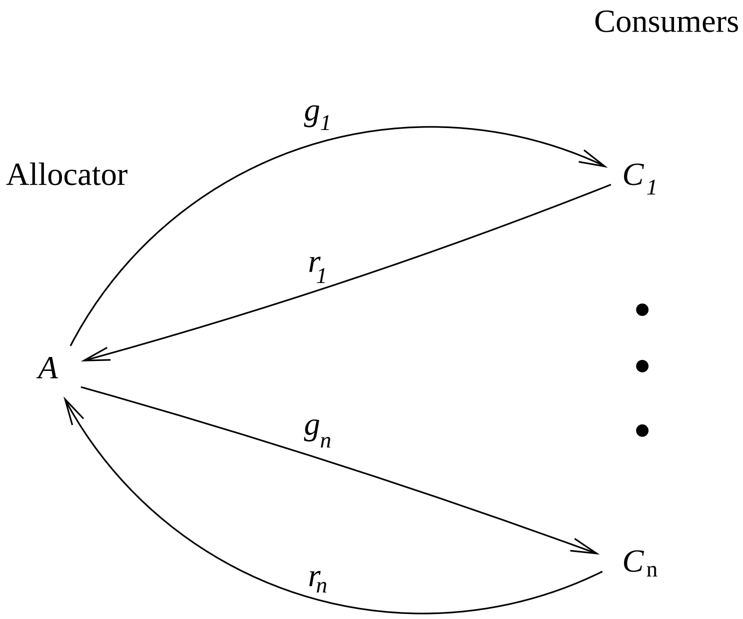
- fault freedom

$$\square[at\_m_3 \;\rightarrow\; [y_2 \ne 0 \;\wedge\; (b \bmod y_2)=0]]$$

- mutual exclusion

$$\square[\neg(at\_\ell_{2..4} \;\wedge\; at\_m_{3..5})]$$

Example 2: Resource-Allocator Program



C_i requests	—	$r_i :=$ T
A grants	—	$g_i :=$ T
C_i releases	—	$r_i :=$ F
A acknowledges	—	$g_i :=$ F

Properties

- mutual exclusion

$$\square(\textstyle\sum g_i \le 1)$$

$1 \rightarrow$ T, $\quad 0 \rightarrow$ F

- conformance with protocol

$$\begin{array}{rcl} (\neg g_i) & \Rightarrow & (\neg g_i)\;\mathcal{W}\;(\neg g_i \wedge r_i) \\ r_i & \Rightarrow & r_i\;\mathcal{W}\;(r_i \wedge g_i) \\ g_i & \Rightarrow & g_i\;\mathcal{W}\;(g_i \wedge \neg r_i) \\ (\neg r_i) & \Rightarrow & (\neg r_i)\;\mathcal{W}\;(\neg r_i \wedge \neg g_i) \end{array}$$

- 1-bounded overtaking

$$r_i \;\Rightarrow\; (\neg g_j)\;\mathcal{W}\quad g_j\;\mathcal{W}\;(\neg g_j)\;\mathcal{W}\;g_i$$
for every j, $j \ne i$

- liveness

$$\begin{array}{rcl} r_i & \Rightarrow & \diamondsuit\, g_i \\ g_i & \Rightarrow & \diamondsuit(\neg r_i) \\ (\neg r_i) & \Rightarrow & \diamondsuit(\neg g_i) \end{array}$$

Example 3A: Program PRIME

local y: **integer** **where** $y=1$
ℓ_0 : **loop forever do**

$$\cdots \| \left[\begin{array}{l} \quad\vdots \\ \ell_5 : \;\mathbf{print}(y) \\ \ell_6 : \quad\vdots \end{array}\right] \| \cdots$$

output: 2, 3, 5, 7, 11, 13, $\cdots$

$$\underbrace{\text{printed}(u)\text{:}}_{j} \quad \underbrace{\ominus\, at\_\ell_5}_{j-1} \wedge \underbrace{at\_\ell_6}_{j} \wedge \underbrace{y=u}_{j}$$

Why $\ominus\, at\_\ell_5$?

- **only primes**
  $\forall u.\;\; \text{printed}(u) \;\Rightarrow\; \text{prime}(u)$
- **all primes**
  $\forall u.\;\; \text{prime}(u) \;\rightarrow\; \diamondsuit\,\text{printed}(u)$
- **monotonicity**
  $\forall u, u'.\;\; \text{printed}(u) \;\Rightarrow$
  $\widehat{\square}(\text{printed}(u') \;\rightarrow\; u' > u)$

where u,u' are rigid

Asynchronous Communication

sending event (predicate)

$$[\alpha \prec v]: \quad \neg \mathit{first} \ \wedge \ \alpha = \alpha^- \bullet v$$

"The value v (of e) has just been sent to α"

$$\begin{array}{ccccc} \alpha & \alpha \Leftarrow e & [\alpha \prec v] & & [\alpha \not\prec v] \\ j-1 & \text{———} & j & \text{———} & j+1 \end{array}$$

receiving event

$$[\alpha \succ v]: \quad \neg \mathit{first} \ \wedge \ v \bullet \alpha = \alpha^-$$

"The value v has just been received (in u)
from channel α"

$$\begin{array}{ccccc} \alpha & \alpha \Rightarrow u & [\alpha \succ v] & & [\alpha \not\succ v] \\ j-1 & \text{———} & j & \text{———} & j+1 \end{array}$$

Synchronous Communication

$$[\alpha \prec\!\succ v]$$

$$\begin{array}{ccccc} \alpha & \begin{array}{c} \ell\colon\ \alpha \Leftarrow e : \widehat{\ell} \\ m\colon\ \alpha \Rightarrow u : \widehat{m} \end{array} & [\alpha \prec\!\succ v] & & [\alpha \not\prec\!\succ v] \\ j-1 & \text{———} & j & \text{———} & j+1 \end{array}$$

$$[\alpha \prec\!\succ v]: \ \bigvee_{\langle \ell, m \rangle} \begin{bmatrix} \{[\ell], [m]\} \subseteq \pi^- \quad \wedge \\ \pi \ = \ (\pi^- - \{[\ell], [m]\}) \cup \{[\widehat{\ell}], [\widehat{m}]\} \wedge \\ v = e^- \end{bmatrix}$$

"A synchronous communication has just taken place"

Note: $\bigvee$ ranges over all pairs of parallel $\Rightarrow, \Leftarrow$ statements on α.

Example 3B: Program PRIME

$$\begin{array}{ll} \ell_0: & \textbf{loop forever do} \\ & \cdots \| \begin{bmatrix} \vdots \\ \gamma \Leftarrow x \\ \vdots \end{bmatrix} \| \cdots \end{array}$$

output channel γ: 2, 3, 5, 7, 11, 13, $\cdots$

$$\begin{array}{ccccc} & \gamma \Leftarrow x & [\gamma \prec p] & & [\gamma \not\prec p] \\ j-1 & \text{———} & j & \text{———} & j+1 \end{array}$$

- **only primes**
  $\forall u. \quad [\gamma \prec u] \ \Rightarrow \ \text{prime}(u)$
- **all primes**
  $\forall u. \quad \text{prime}(u) \ \rightarrow \ \diamondsuit[\gamma \prec u]$
- **monotonicity**
  $\forall m, m'. \ \ ([\gamma \prec m] \ \wedge \ \widehat{\diamondminus}[\gamma \prec m']) \ \Rightarrow \ m' < m$

Why $\widehat{\diamondminus}$ and not $\diamondminus$?

Verification: Motivating Example

Peterson's Algorithm
(for mutual exclusion)
Version 1 – INCORRECT

$$\textbf{local} \ \ y_1, y_2\colon \ \textbf{boolean} \ \ \textbf{where} \ y_1 = \text{F}, y_2 = \text{F}$$

$$P_1 :: \quad \begin{array}{ll} \ell_0: & \textbf{loop forever do} \\ & \begin{bmatrix} \ell_1: & \textbf{noncritical} \\ \ell_2: & y_1 := \text{T} \\ \ell_3: & \textbf{await} \ \neg y_2 \\ \ell_4: & \textbf{critical} \\ \ell_5: & y_1 := \text{F} \end{bmatrix} \end{array}$$

$$\|$$

$$P_2 :: \quad \begin{array}{ll} m_0: & \textbf{loop forever do} \\ & \begin{bmatrix} m_1: & \textbf{noncritical} \\ m_2: & y_2 := \text{T} \\ m_3: & \textbf{await} \ \neg y_1 \\ m_4: & \textbf{critical} \\ m_5: & y_2 := \text{F} \end{bmatrix} \end{array}$$

Peterson's Algorithm (Con't)

- protection variables: y_1, y_2

$y_1 = \text{F}$, $y_2 = \text{F}$	—	initially
$\ell_2 :\ y_1 := \text{T}$	—	P_1 is interested
$\ell_3 :$ **await** $(\neg y_2) \vee \ldots$	—	P_1 waits
$\ell_5 :\ y_1 := \text{F}$	—	P_1 resets y_1

Problem: potential deadlock
may reach ℓ_3, m_3 with $y_1 = y_2 = \text{T}$

$$\ldots \to \langle\{\ell_2, m_2\}, \overset{y_1}{\text{F}}, \overset{y_2}{\text{F}}\rangle \overset{\ell_2}{\to} \langle\{\ell_3, m_2\}, \text{T}, \text{F}\rangle$$
$$\overset{m_2}{\to} \langle\{\ell_3, m_3\}, \text{T}, \text{T}\rangle \overset{\tau_I}{\to} \ldots$$

Fix: add signature variable s

$\ell_2 :\ s := 1$	—	P_1 requests priority
$\ell_3 :$ **await** $\ldots \vee (s \underset{\uparrow}{=} 2)$	—	P_1 has priority

P_2 was the last to request priority

Peterson's Algorithm
Version 2 (signature variable) – CORRECT

local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$P_1 ::$ $\ell_0 :$ **loop forever do**
$$\left[\begin{array}{ll} \ell_1 : & \textbf{noncritical} \\ \ell_2 : & (y_1,\ s) := (\text{T},\ 1) \\ \ell_3 : & \textbf{await } (\neg y_2) \vee (s = 2) \\ \ell_4 : & \textbf{critical} \\ \ell_5 : & y_1 := \text{F} \end{array}\right]$$

||

$P_2 ::$ $m_0 :$ **loop forever do**
$$\left[\begin{array}{ll} m_1 : & \textbf{noncritical} \\ m_2 : & (y_2,\ s) := (\text{T},\ 2) \\ m_3 : & \textbf{await } (\neg y_1) \vee (s = 1) \\ m_4 : & \textbf{critical} \\ m_5 : & y_2 := \text{F} \end{array}\right]$$

Peterson's Algorithm
Properties of version 2

Mutual Exclusion

$$\Box \neg(at\_\ell_4 \wedge at\_m_4)$$

1-Bounded Overtaking for P_1

$$at\_\ell_3 \Rightarrow (\neg at\_m_4)\, \mathcal{W}\, at\_m_4\, \mathcal{W}\, (\neg at\_m_4)\, \mathcal{W}\, at\_\ell_4$$

Accessibility for P_1

$$at\_\ell_2 \Rightarrow \Diamond\, at\_\ell_4$$

Peterson's Algorithm
Version 3 (split assignments) – INCORRECT

local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$P_1 ::$ $\ell_0 :$ **loop forever do**
$$\left[\begin{array}{ll} \ell_1 : & \textbf{noncritical} \\ \boxed{\begin{array}{ll} \ell_2 : & s := 1 \\ \ell_3 : & y_1 := \text{T} \end{array}} & \\ \ell_4 : & \textbf{await } (\neg y_2) \vee (s = 2) \\ \ell_5 : & \textbf{critical} \\ \ell_6 : & y_1 := \text{F} \end{array}\right]$$

||

$P_2 ::$ $m_0 :$ **loop forever do**
$$\left[\begin{array}{ll} m_1 : & \textbf{noncritical} \\ \boxed{\begin{array}{ll} m_2 : & s := 2 \\ m_3 : & y_2 := \text{T} \end{array}} & \\ m_4 : & \textbf{await } (\neg y_1) \vee (s = 1) \\ m_5 : & \textbf{critical} \\ m_6 : & y_2 := \text{F} \end{array}\right]$$

$$(y_1, s) := (\text{T}, 1) \quad \rightarrow \quad \boxed{\begin{array}{ll} \ell_2\text{:} & s := 1 \\ \ell_3\text{:} & y_1 := \text{T} \end{array}}$$

$$(y_2, s) := (\text{T}, 2) \quad \rightarrow \quad \boxed{\begin{array}{ll} m_2\text{:} & s := 2 \\ m_3\text{:} & y_2 := \text{T} \end{array}}$$

Problem: violation of mutual exclusion

$$\begin{array}{llll} \ldots \rightarrow & \langle\{\ell_3, m_3\}, \overset{s}{2}, \overset{y_1}{\text{F}}, \overset{y_2}{\text{F}}\rangle & \overset{m_3}{\rightarrow} & \langle\{\ell_3, m_4\}, 2, \text{F}, \text{T}\rangle \\ \overset{m_4}{\rightarrow} & \langle\{\ell_3, m_5\}, 2, \text{F}, \text{T}\rangle & \overset{\ell_3}{\rightarrow} & \langle\{\ell_4, m_5\}, 2, \text{T}, \text{T}\rangle \\ \overset{\ell_4}{\rightarrow} & \langle\{\ell_5, m_5\}, 2, \text{T}, \text{T}\rangle & \rightarrow & \ldots \end{array}$$

Fix: reverse the statements

$$(y_1, s) := (\text{T}, 1) \quad \rightarrow \quad \boxed{\begin{array}{ll} \ell_2\text{:} & y_1 := \text{T} \\ \ell_3\text{:} & s := 1 \end{array}}$$

$$(y_2, s) := (\text{T}, 2) \quad \rightarrow \quad \boxed{\begin{array}{ll} m_2\text{:} & y_2 := \text{T} \\ m_3\text{:} & s := 2 \end{array}}$$

Peterson's Algorithm
Version 4 (reverse statements)
CORRECT???

$$\begin{array}{llll} \textbf{local} & y_1, y_2\text{:} & \textbf{boolean} & \textbf{where } y_1 = \text{F}, y_2 = \text{F} \\ & s \quad : & \textbf{integer} & \textbf{where } s = 1 \end{array}$$

$$\ell_0: \ \textbf{loop forever do}$$

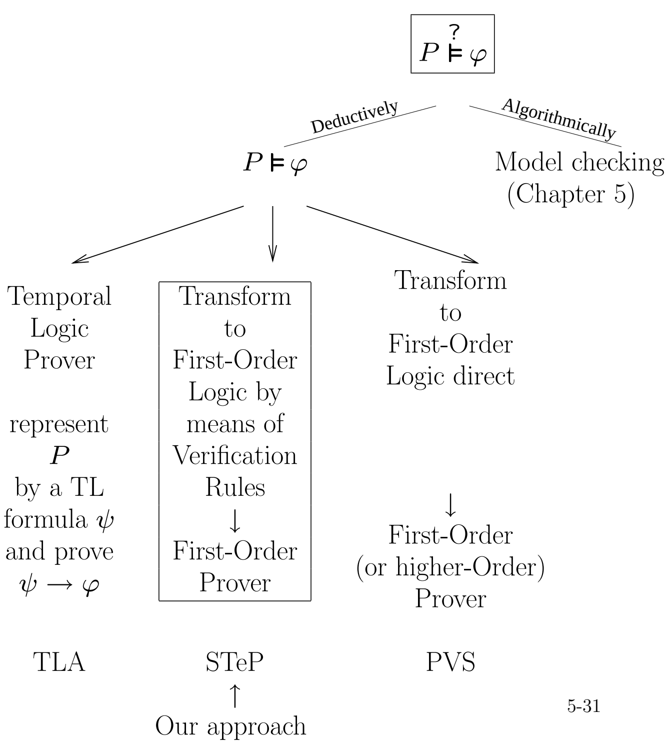
$$P_1 :: \left[\begin{array}{ll} \ell_1: & \textbf{noncritical} \\ \ell_2: & y_1 := \text{T} \\ \ell_3: & s := 1 \\ \ell_4: & \textbf{await } (\neg y_2) \vee (s = 2) \\ \ell_5: & \textbf{critical} \\ \ell_6: & y_1 := \text{F} \end{array}\right]$$

$||$

$$m_0: \ \textbf{loop forever do}$$

$$P_2 :: \left[\begin{array}{ll} m_1: & \textbf{noncritical} \\ m_2: & y_2 := \text{T} \\ m_3: & s := 2 \\ m_4: & \textbf{await } (\neg y_1) \vee (s = 1) \\ m_5: & \textbf{critical} \\ m_6: & y_2 := \text{F} \end{array}\right]$$

Proving temporal properties of reactive systems



Proving temporal properties of reactive systems (Con't)

Textbook:	Proof methods for safety properties: $P \vDash \Box\, \varphi$, where φ is a past formula (no future operators)
Chapters 1, 2:	Proof methods for invariance properties: $P \vDash \Box\, q$, where q is a state formula (no temporal operators)
Vol III of textbook	Proof methods for progress properties.