CS156: The Calculus of Computation

Zohar Manna
Winter 2010

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Calculus of Computation?

> *It is reasonable to hope that the relationship between* **computation** *and* **mathematical logic** *will be as fruitful in the next century as that between* **analysis** *and* **physics** *in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy
A Basis for a Mathematical Theory of Computation, 1963

Grading

- Homeworks (40%)
  - weekly (totally 8)
  - no late assignments
  - no collaboration
- Final Exam (60%)
  - open book and notes
  - date: Monday, March 15th, 8:30-11:30 a.m.

Coverage

- Skip * sections
- Skip Chapter 6 and 12 of the book
- Skip complexity remarks

Website

- `cs156.stanford.edu`

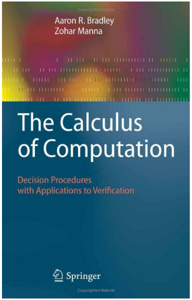
Textbook

THE CALCULUS OF COMPUTATION:
Decision Procedures with
Applications to Verification

by
Aaron Bradley
Zohar Manna

Springer 2007

There are two copies in CS-Math Library and you could also use `socrates.stanford.edu` to read the book according to its policy.



Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures

Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
   Inductive assertion method, Ranking function method

Part II: Decision Procedures

7. Quantified Linear Arithmetic
   Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
   Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
    Nelson-Oppen combination method
11. Arrays
    More than quantifier-free fragment

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Motivation

Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- in software verification.
- in hardware verification

Consider the following program:

```
for
  @ ℓ ≤ i ≤ u ∧ (rv ↔ ∃j. ℓ ≤ j < i ∧ a[j] = e)
  (int i := ℓ; i ≤ u; i := i + 1) {
  if (a[i] = e) rv := true;
}
```

How can we decide whether the formula is a loop invariant?

Motivation II

Prove:

```
assume ℓ ≤ i ≤ u ∧ (rv ↔ ∃j. ℓ ≤ j < i ∧ a[j] = e)
assume i ≤ u
assume a[i] = e
rv := true;
i := i + 1
@ ℓ ≤ i ≤ u ∧ (rv ↔ ∃j. ℓ ≤ j < i ∧ a[j] = e)
```


Motivation III

```
assume ℓ ≤ i ≤ u ∧ (rv ↔ ∃j. ℓ ≤ j < i ∧ a[j] = e)
assume i ≤ u
assume a[i] ≠ e
i := i + 1
@ ℓ ≤ i ≤ u ∧ (rv ↔ ∃j. ℓ ≤ j < i ∧ a[j] = e)
```

A Hoare triple {P} S {Q} holds, iff

$$P \rightarrow wp(S, Q)$$

(wp denotes "weakest precondition")

Motivation IV

For assignments wp is computed by substitution:

---

assume $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e)$

assume $i \leq u$

assume $a[i] = e$

rv := true;

$i := i + 1$

@ $\ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e)$

Substituting $\top$ for rv and $i+1$ for i, the postcondition (denoted by the @ symbol) holds if and only if:

$$\begin{array}{l} \ell \leq i \leq u \wedge (rv \leftrightarrow \exists j.\ \ell \leq j < i \wedge a[j] = e) \wedge i \leq u \wedge a[i] = e \\ \rightarrow \ell \leq i+1 \leq u \wedge (\top \leftrightarrow \exists j.\ \ell \leq j < i+1 \wedge a[j] = e) \end{array}$$

Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \mathit{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

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Chapter 1: Propositional Logic (PL)

Propositional Logic (PL)

PL Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false")		
	propositional variables $P, Q, R, P_1, Q_1, R_1, \ldots$		
Literal	atom α or its negation $\neg\alpha$		
Formula	literal or application of a logical connective to formulae F, F_1, F_2		
	$\neg F$	"not"	(negation)
	$F_1 \wedge F_2$	"and"	(conjunction)
	$F_1 \vee F_2$	"or"	(disjunction)
	$F_1 \rightarrow F_2$	"implies"	(implication)
	$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Example:

formula $F : (P \wedge Q) \rightarrow (\top \vee \neg Q)$
atoms: P, Q, $\top$
literals: P, Q, $\top$, $\neg Q$
subformulae: P, Q, $\top$, $\neg Q$, $P \wedge Q$, $\top \vee \neg Q$, F
abbreviation
$F : P \wedge Q \rightarrow \top \vee \neg Q$

PL Semantics (meaning of PL)

Formula F + Interpretation I = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of F under I:

F	$\neg F$
0	1
1	0

where 0 corresponds to value false
1 true

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$F : P \wedge Q \rightarrow P \vee \neg Q$
$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

1 = true 0 = false

F evaluates to true under I; i.e., $I[F] = \text{true}$.

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I
$I \not\models F$ false

Base Case:

$I \models \top$ $I \not\models \bot$
$I \models P$ iff $I[P] = \text{true}$; i.e., P is true under I
$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$
$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$
$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$ (or both)
$I \models F_1 \rightarrow F_2$ iff $I \models F_1$ implies $I \models F_2$
$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Note:

$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.
$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.
$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

Example of Inductive Reasoning:

$$F: \; P \wedge Q \;\rightarrow\; P \vee \neg Q$$

$$I: \; \{P \;\mapsto\; \text{true}, \; Q \;\mapsto\; \text{false}\}$$

1.	I	$\models$	P	since $I[P] = \text{true}$	
2.	I	$\not\models$	Q	since $I[Q] = \text{false}$	
3.	I	$\models$	$\neg Q$	by 2 and $\neg$	
4.	I	$\not\models$	$P \wedge Q$	by 2 and $\wedge$	
5.	I	$\models$	$P \vee \neg Q$	by 1 and $\vee$	
6.	I	$\models$	F	by 4 and $\rightarrow$	Why?

Thus, F is true under I.
Note: steps 1, 3, and 5 are nonessential.

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.
F valid iff for all interpretations I, $I \models F$.

F is valid iff $\neg F$ is unsatisfiable

Goal: devise an algorithm to decide validity or unsatisfiability of formula F.

Method 1: Truth Tables

Example $F: P \wedge Q \;\rightarrow\; P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F: P \vee Q \;\rightarrow\; P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F	
0	0	0	0	1	$\leftarrow$ satisfying I
0	1	1	0	0	$\leftarrow$ falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

- Assume F is not valid and I a falsifying interpretation: $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable, F is invalid.
- If in every branch of proof a contradiction reached, F is valid.

Proof Rules for Semantic Arguments I

$$\begin{array}{c} I \models \neg F \\ \hline I \not\models F \end{array} \qquad\qquad \begin{array}{c} I \not\models \neg F \\ \hline I \models F \end{array}$$

$$\begin{array}{c} I \models F \wedge G \\ \hline I \models F \\ I \models G \end{array} \leftarrow \text{and} \qquad\qquad \begin{array}{c} I \not\models F \wedge G \\ \hline I \not\models F \mid I \not\models G \end{array}\ \nwarrow \text{or}$$

$$\begin{array}{c} I \models F \vee G \\ \hline I \models F \mid I \models G \end{array} \qquad\qquad \begin{array}{c} I \not\models F \vee G \\ \hline I \not\models F \\ I \not\models G \end{array}$$

Proof Rules for Semantic Arguments II

$$\begin{array}{c} I \models F \rightarrow G \\ \hline I \not\models F \mid I \models G \end{array} \qquad\qquad \begin{array}{c} I \not\models F \rightarrow G \\ \hline I \models F \\ I \not\models G \end{array}$$

$$\begin{array}{c} I \models F \leftrightarrow G \\ \hline I \models F \wedge G \mid I \not\models F \vee G \end{array} \qquad\qquad \begin{array}{c} I \not\models F \leftrightarrow G \\ \hline I \models F \wedge \neg G \mid I \models \neg F \wedge G \end{array}$$

$$\begin{array}{c} I \models F \\ I \not\models F \\ \hline I \models \bot \end{array}$$

Example: Prove

$$F:\ P \wedge Q \ \rightarrow\ P \vee \neg Q \quad \text{is valid.}$$

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$P \wedge Q \rightarrow P \vee \neg Q$	assumption
2.	I	$\models$	$P \wedge Q$	1 and $\rightarrow$
3.	I	$\not\models$	$P \vee \neg Q$	1 and $\rightarrow$
4.	I	$\models$	P	2 and $\wedge$
5.	I	$\not\models$	P	3 and $\vee$
6.	I	$\models$	$\bot$	4 and 5 are contradictory

Thus F is valid.

Example: Prove

$$F:\ (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R) \quad \text{is valid.}$$

Let's assume that F is not valid.

1.	$I \not\models F$	assumption
2.	$I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$	1 and $\rightarrow$
3.	$I \not\models P \rightarrow R$	1 and $\rightarrow$
4.	$I \models P$	3 and $\rightarrow$
5.	$I \not\models R$	3 and $\rightarrow$
6.	$I \models P \rightarrow Q$	2 and $\wedge$
7.	$I \models Q \rightarrow R$	2 and $\wedge$

6.	$I \models P \rightarrow Q$	2 and $\wedge$
7.	$I \models Q \rightarrow R$	2 and $\wedge$
8a.	$I \not\models P$	6 and $\rightarrow$ (case a)
9a.	$I \models \bot$	4 and 8
8b.	$I \models Q$	6 and $\rightarrow$ (case b)
9ba.	$I \not\models Q$	7 and $\rightarrow$ (subcase ba)
10ba.	$I \models \bot$	8b and 9ba
9bb.	$I \models R$	7 and $\rightarrow$ (subcase bb)
10bb.	$I \models \bot$	5 and 9bb
9b.	$I \models \bot$	10ba and 10bb
8.	$I \models \bot$	9a and 9b

Our assumption is contradictory in all cases, so F is valid.

Example 3: Is

$$F: P \vee Q \rightarrow P \wedge Q$$

valid? Assume F is not valid:

1.	$I \not\models P \vee Q \rightarrow P \wedge Q$	assumption
2.	$I \models P \vee Q$	1 and $\rightarrow$
3.	$I \not\models P \wedge Q$	1 and $\rightarrow$
4a.	$I \models P$	2, $\vee$ (case a)
5aa.	$I \not\models P$	3, $\vee$ (subcase aa)
6aa.	$I \models \bot$	4a, 5aa
5ab.	$I \not\models Q$	3, $\vee$ (subcase ab)
6ab.	?	
5a.	?	

4b.	$I \models Q$	2, $\vee$ (case b)
5ba.	$I \not\models P$	3, $\vee$ (subcase ba)
6ba.	?	
5bb.	$I \not\models Q$	3, $\vee$ (subcase bb)
6bb.	$I \models \bot$	4b, 5bb
5b.	?	
5.	?	

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that F is valid. To demonstrate that F is not valid, however, we must find a falsifying interpretation (here are two):

$I_1: \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ $\quad I_2: \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

Note: we have to derive a contradiction in all cases for F to be valid!

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid, that is, both $F_1 \rightarrow F_2$ and $F_2 \rightarrow F_1$ are valid.

F_1 entails F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations I, $I \models F_1 \rightarrow F_2$

Note: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

Example: Show

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

i.e.

$$F : (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \textit{ is valid}.$$

Assume F is not valid, then we have two cases:

Case a: $I \not\models \neg P \vee Q$,
$I \models P \rightarrow Q$

Case b: $I \models \neg P \vee Q$,
$I \not\models P \rightarrow Q$

Derive contradictions in both cases.

Normal Forms

1. Negation Normal Form (NNF)

   $\neg, \wedge, \vee$ are the only boolean connectives allowed.

   Negations may occur only in literals of the form $\neg P$.

   To transform F into equivalent F' in NNF, apply the following template equivalences recursively (and left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l} \neg(F_1 \wedge F_2) \Leftrightarrow \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \Leftrightarrow \neg F_1 \wedge \neg F_2 \end{array}\right\} \text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$$

   "Complete" syntactic restriction: every F has an equivalent F' in NNF.

Example: Convert

$$F : \neg(P \rightarrow \neg(P \wedge Q))$$

to NNF.

$$\begin{array}{lll} F' & : \neg(\neg P \vee \neg(P \wedge Q)) & \rightarrow \\ F'' & : \neg\neg P \wedge \neg\neg(P \wedge Q) & \text{De Morgan's Law} \\ F''' & : P \wedge P \wedge Q & \neg\neg \end{array}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)

   Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

   To convert F into equivalent F' in DNF,
   transform F into NNF and then
   use the following template equivalences (left-to-right):

$$\left.\begin{array}{lll} (F_1 \vee F_2) \wedge F_3 & \Leftrightarrow & (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) & \Leftrightarrow & (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{array}\right\} \textit{dist}$$

   Note: formulae can grow exponentially as the distributivity laws are applied.

Example: Convert

$$F: \; (Q_1 \lor \neg\neg Q_2) \land (\neg R_1 \; \rightarrow \; R_2)$$

into equivalent DNF

$$\begin{array}{lll} F' : (Q_1 \lor Q_2) \land (R_1 \lor R_2) & & \text{in NNF} \\ F'' : (Q_1 \land (R_1 \lor R_2)) \lor (Q_2 \land (R_1 \lor R_2)) & & \text{dist} \\ F''' : (Q_1 \land R_1) \lor (Q_1 \land R_2) \lor (Q_2 \land R_1) \lor (Q_2 \land R_2) & & \text{dist} \end{array}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{rcl} (F_1 \land F_2) \lor F_3 & \Leftrightarrow & (F_1 \lor F_3) \land (F_2 \lor F_3) \\ F_1 \lor (F_2 \land F_3) & \Leftrightarrow & (F_1 \lor F_2) \land (F_1 \lor F_3) \end{array}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : P \; \leftrightarrow \; (Q \; \rightarrow \; R)$$

to an equivalent formula F' in CNF.

First get rid of $\leftrightarrow$:

$$F_1 : (P \; \rightarrow \; (Q \; \rightarrow \; R)) \land ((Q \; \rightarrow \; R) \; \rightarrow \; P)$$

Now replace $\rightarrow$ with $\lor$:

$$F_2 : (\neg P \lor (\neg Q \lor R)) \land (\neg(\neg Q \lor R) \lor P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \lor \neg Q \lor R) \land ((\neg\neg Q \land \neg R) \lor P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \lor \neg Q \lor R) \land ((Q \land \neg R) \lor P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \lor \neg Q \lor R) \land (Q \lor P) \land (\neg R \lor P)$$

Equisatisfiability

Definition

F and F' are *equisatisfiable*, iff

F is satisfiable if and only if F' is satisfiable

Every formula is equisatifiable to either $\top$ or $\bot$.

Goal: Decide satisfiability of PL formula F

- Step 1: Convert F to equisatisfiable formula F' in CNF
- Step 2: Decide satisfiability of formula F' in CNF

Step 1: Convert F to equisatisfiable formula F' in CNF I

There is an *efficient conversion* of F to F' where

- F' is in CNF and
- F and F' are equisatisfiable

Note: efficient means polynomial in the size of F.

Basic Idea:

- Introduce a new variable P_G for every subformula G of F, unless G is already an atom.

Step 1: Convert F to equisatisfiable formula F' in CNF II

- For each subformula

$$G : \; G_1 \circ G_2,$$

produce a small formula

$$P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2}.$$

Here $\circ$ denotes an arbitrary connective ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$); if the connective is $\neg$, G_1 should be ignored.

Step 1: Convert F to equisatisfiable formula F' in CNF III

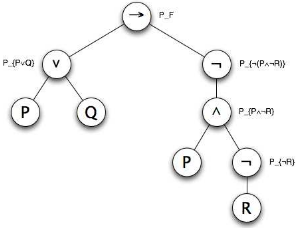

Figure: Parse tree for $F : P \vee Q \;\rightarrow\; \neg(P \wedge \neg R)$

Step 1: Convert F to equisatisfiable formula F' in CNF IV

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\text{CNF}(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2}) \; .$$

Let S_F be the set of all non-atom subformulae G of F (including F itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} CNF(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F. (Why?)

The number of subformulae is linear in the size of F.
The time to convert one small formula is constant!

Example: CNF I

Convert

$$F:\ P \vee Q \ \rightarrow\ P \wedge \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: P_F, $P_{P\vee Q}$, $P_{P\wedge\neg R}$, $P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \text{CNF}(P_F \ \leftrightarrow\ (P_{P\vee Q} \ \rightarrow\ P_{P\wedge\neg R}))$:

$$(\neg P_F \vee \neg P_{P\vee Q} \vee P_{P\wedge\neg R}) \wedge (P_F \vee P_{P\vee Q}) \wedge (P_F \vee \neg P_{P\wedge\neg R})$$

- $F_2 = \text{CNF}(P_{P\vee Q} \ \leftrightarrow\ P \vee Q)$:

$$(\neg P_{P\vee Q} \vee P \vee Q) \wedge (P_{P\vee Q} \vee \neg P) \wedge (P_{P\vee Q} \vee \neg Q)$$

Example: CNF II

- $F_3 = \text{CNF}(P_{P\wedge\neg R} \ \leftrightarrow\ P \wedge P_{\neg R})$:

$$(\neg P_{P\wedge\neg R} \vee P) \wedge (\neg P_{P\wedge\neg R} \vee P_{\neg R}) \wedge (P_{P\wedge\neg R} \vee \neg P \vee \neg P_{\neg R})$$

- $F_4 = \text{CNF}(P_{\neg R} \ \leftrightarrow\ \neg R)$:

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to F.

Step 2: Decide the satisfiability of PL formula F' in CNF

<u>Boolean Constraint Propagation (BCP)</u>

If a clause contains one literal ℓ,

Set ℓ to $\top$:	$\cdots \wedge \ell \wedge \cdots$ ($\ell \to \top$)
Remove all clauses containing ℓ:	$\cdots \wedge \cancel{(\cdots \vee \ell \vee \cdots)} \wedge \cdots$
Remove $\neg\ell$ in all clauses:	$\cdots \wedge (\cdots \vee \cancel{\neg\ell} \vee \cdots) \wedge \cdots$

based on the unit resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \quad \leftarrow \text{clause}$$

<u>Pure Literal Propagation (PLP)</u>

If P occurs only positive (without negation), set it to $\top$.
If P occurs only negative set it to $\bot$.
Then do the simplifications as in Boolean Constraint Propagation

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

<u>Decision Procedure DPLL</u>: Given F in CNF

```
let rec DPLL F =
  let F' = BCP F in
  let F'' = PLP F' in
  if F'' = ⊤ then true
  else if F'' = ⊥ then false
  else
     let P = CHOOSE vars(F'') in
     (DPLL F''{P ↦ ⊤}) ∨ (DPLL F''{P ↦ ⊥})
```

Simplification

Simplify according to the template equivalences (left-to-right) [exercise 1.2]

$$\begin{array}{ccc} \neg\bot \Leftrightarrow \top & \neg\top \Leftrightarrow \bot & \neg\neg F \Leftrightarrow F \\ F \wedge \top \Leftrightarrow F & F \wedge \bot \Leftrightarrow \bot & \cdots \\ F \vee \top \Leftrightarrow \top & F \vee \bot \Leftrightarrow F & \cdots \end{array}$$

Example I

Consider

$$F: (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

Branching on Q

On the first branch, we have

$$F\{Q \mapsto \top\}: (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \quad (\neg R)}{\bot},$$

so $F\{Q \mapsto \top\} = \bot \Rightarrow$ false.

Example II

Recall

$$F: (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

On the other branch, we have

$$F\{Q \mapsto \bot\}: (\neg P \vee R).$$

Furthermore, by PLP,

$$F\{Q \mapsto \bot,\ R \mapsto \top,\ P \mapsto \bot\} = \top \Rightarrow \text{true}$$

Thus F is satisfiable with satisfying interpretation

$$I: \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}.$$

Example

$$F: (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$